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2001 J. Phys. A: Math. Gen. 34 157

(http://iopscience.iop.org/0305-4470/34/1/312)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 157–164 www.iop.org/Journals/ja PII: S0305-4470(01)14552-X

q-deformed solitons and quantum solitons of the Maxwell–Bloch lattice

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Received 7 June 2000

Abstract

We report for the first time exact solutions of a completely integrable nonlinear lattice system for which the dynamical variables satisfy a q-deformed Lie algebra—the Lie–Poisson algebra $su_q(2)$. The system considered is a q-deformed lattice for which in the continuum limit the equations of motion become the envelope Maxwell–Bloch (or SIT) equations describing the resonant interaction of light with a nonlinear dielectric. Thus the N-soliton solutions we report here are the natural q-deformations, necessary for a lattice, of the well known multi-soliton and breather solutions of self-induced transparency (SIT). The method we use to find these solutions is a generalization of the Darboux–Bäcklund dressing method. The extension of these results to quantum solitons is sketched.

PACS numbers: 0545Y, 0545M, 4265T

1. Introduction

The Maxwell–Bloch (MB) system of equations has been fundamental to much of theoretical quantum optics and nonlinear optics since they were first introduced in the late 1960s (some history of the subject is given in [1] and also in [2]). These MB systems are of abiding theoretical interest. A feature is that their complete integrability is handed down from the 'reduced MB' or (RMB) equations to the envelope MB (or self-induced transparency (SIT)) equations, thence, at resonance, to the sine–Gordon equation (cf, e.g., [3, 4]). Each member of this hierarchy has important physical applications, while the physics of SIT, in particular, remains a very active field of current research [5], even into the femto second pulse regime [1, 6].

Our recent paper [1] followed up ideas of quantum groups and their relevance to integrable systems theory and derived a q-deformed lattice version of the envelope MB system together with its zero-curvature representation: in the continuum limit these lattice equations become the resonant envelope MB (or SIT) equations. In this paper we now report exact N-soliton solutions of this q-deformed dynamical system. Solitons of the lattice equations were promised in [1], and a Riemann–Hilbert method of solution sketched. However, for the pure N-soliton solution reported in this paper it is more convenient to use a variant of the Darboux–Bäcklund dressing method which (see below) makes an ansatz for the dressing in terms of appropriate N bound states eigenvalues.

Historically [3, 7] multi-soliton solutions of the SIT equations were found by the method which become Hirota's method [2]; Lamb [8] gave the inverse scattering solutions; the inverse method for the RMB equations was used in [3, 9] to obtain the multi-soliton solutions for the SIT equations; [10] gave a further account of inverse scattering for these SIT equations. These several results on inverse scattering confirmed the generality of a method first devised to solve the Korteweg–de Vries equation [2].

Expressed in terms of the complex slowly varying envelopes for the electric field and polarization ε, ρ and the real inversion $\mathcal{N}$, the SIT equations can be put in the form (e.g. [10]),

$$\begin{aligned} &\partial_\xi \varepsilon = \langle \rho \rangle \\ &\partial_\tau \rho + 2\mathrm{i}\eta\rho = \mathcal{N}\varepsilon \\ &\partial_\tau \mathcal{N} = -\tfrac{1}{2}(\varepsilon^*\rho + \varepsilon\rho^*). \end{aligned} \tag{1}$$

Propagation is along a coordinate z and $\xi = \Omega x$ with $x = z/c$. The time τ is a retarded time, $\tau = \Omega(t - x)$; $\eta = (\omega - \omega_0)/2\Omega$ is the detuning and $\Omega = 2\pi n_0\omega_0\mu^2/\hbar$ is the coupling constant. The number n_0 is the density of two-level atoms with the non-degenerate transition frequency ω_0 (rad s^{-1}); μ is the matrix element for dipole-allowed transitions at ω_0. The star denotes complex conjugation and $\langle\cdot\rangle = \int_{-\infty}^{\infty} h(\eta)(\cdot)\,\mathrm{d}\eta$ is the average over inhomogeneous broadening: $h(\eta)$ is a δ-function in the sharp-line limit case [3].

In [1] we constructed the completely integrable lattice system whose equations of motion for three dynamical variables s_n, H_n and β_n at each lattice site n can be written as

$$\begin{aligned} &\partial_t \beta_n = -\tfrac{1}{2}q^{2(N_n+H_n)}(\beta_{n+1} + \beta_n) - \tfrac{1}{2}\mathrm{i}q^{2N_n}(s_n + s_{n-1}) \\ &\partial_t s_n = -\tfrac{1}{2}\mathrm{i}(\beta_n + \beta_{n+1})(1 + 2\gamma s_n s_n^*) + \tfrac{1}{2}q^{2(N_n+H_n)}(s_n + s_{n-1}) \\ &\partial_t H_n = \tfrac{1}{2}\mathrm{i}(s_n - \mathrm{i}q^{2H_n}\beta_n)(\beta_n^* + \beta_{n+1}^*) - \tfrac{1}{2}\mathrm{i}(s_n^* + \mathrm{i}q^{2H_n}\beta_n^*)(\beta_n + \beta_{n+1}). \end{aligned} \tag{2}$$

Here $q^{2N_n} = 1 + 2\gamma\beta_n^*\beta_n$, $q = \mathrm{e}^\gamma$ and $\gamma > 0$, is a real parameter (a coupling constant, see below). Reference to [1] shows that in equations (2) we use $s_n = \sqrt{2\gamma}\,\chi_n + \mathrm{i}q^{2H_n}\beta_n$: in [1] the second equation is for $\partial_t\chi_n$. As can be checked (and cf [1]) when the lattice spacing $\Delta \to 0$ for a continuum limit with

$$\begin{aligned} &t \to t\Delta^{-1} \qquad x = n\Delta \qquad \beta_n = \sqrt{\Delta}\mathcal{E}(x) \qquad \chi_n = \Delta S(x) \\ &H_n = \Delta S^3(x) \qquad \gamma = \kappa\Delta/2 \qquad \kappa > 0. \end{aligned} \tag{3}$$

One reaches the resonant sharp-line form of the envelope MB (or SIT) equations (1) via the definitions

$$\varepsilon(\xi,\tau) = 2\mathcal{E}(x,t) \qquad \rho(\xi,\tau) = -2\mathrm{i}S(x,t) \qquad N(\xi,\tau) = 2S^3(x,t) \tag{4}$$

with $\Omega = \sqrt{\kappa}$. Our use of 'lattice Maxwell–Bloch system (LMB) equations' for equations (2) stems from this fact.

A Hamiltonian for this LMB system is [1]

$$\mathcal{H}^L = \tfrac{1}{2}\sum_{n=1}^{M}\left\{\sqrt{2\gamma}\left[\chi_n^*(\beta_{n+1} + \beta_n) + \chi_n(\beta_{n+1}^* + \beta_n^*)\right] + \mathrm{i}q^{2H_n}\left(\beta_{n+1}^*\beta_n - \beta_{n+1}\beta_n^*\right)\right\}. \tag{5}$$

For $M < \infty$ it would be natural to impose periodic boundary conditions. However, we shall look for lattice soliton solutions and here think of $M \to \infty$ with suitable boundary conditions still to be specified. The Poisson brackets of equation (5) are

$$\{X_n^*, X_m\} = \mathrm{i}\{2H_n\}\delta_{mn} \qquad \{H_n, X_m\} = -\mathrm{i}X_n\delta_{mn} \tag{6}$$

and the quantities X_n^*, X_n and H_n form the $su_q(2)$ Lie–Poisson algebra, by $\{\cdot\}$ we mean $\{x\} = (q^x - q^{-x})/(2\gamma)$. This algebra has a central element $X_n X_n^* + \{H_n\}^2 = \{S\}^2$. The variables X_n^*, X_n enter (2) via $\chi_n = q^{H_n} X_n$ [1]. The variables β_n, β_n^* (the 'electric fields', see equations (3) and (4) above) satisfy the Lie–Poisson q-boson algebra

$$\{\beta_n, \beta_m^*\} = \mathrm{i}q^{2N_n}\delta_{mn} \qquad \{N_n, \beta_m\} = -\mathrm{i}\beta_n\delta_{mn}. \tag{7}$$

2. The q-deformed solitons

In [1] we obtained the zero-curvature representation of the system (2) which means that we constructed an over-determined linear system for a matrix-function $\Psi_n(\zeta, t)$ such that

$$\Psi_{n+1} = L(\zeta|n)\Psi_n \tag{8}$$

$$\partial_t \Psi_n = V(\zeta|n)\Psi_n \tag{9}$$

where

$$V(\zeta|n) = \sum_{j=-2}^{2} \zeta^j V_j(n) \qquad L(\zeta|n) = \frac{q^{-N_n-H_n}}{2\gamma} \sum_{j=-2}^{2} \zeta^j L_j(n). \tag{10}$$

Here

$$V_0(n) = 2\mathrm{i}\gamma(\beta_n s_{n-1}^* + \beta_n^* s_{n-1})\sigma^z \qquad V_{\pm 2} = \mp\tfrac{1}{4}\sigma^z \tag{11}$$

$$V_{+1}(n) = -\frac{\sqrt{2\gamma}}{2}\begin{pmatrix} 0 & \mathrm{i}\beta_n^* \\ s_{n-1} & 0 \end{pmatrix} \qquad V_{-1}(n) = \frac{\sqrt{2\gamma}}{2}\begin{pmatrix} 0 & s_{n-1}^* \\ -\mathrm{i}\beta_n & 0 \end{pmatrix} \tag{12}$$

while

$$L_0(n) = 2\mathrm{i}\gamma\begin{pmatrix} \beta_n s_n^* & 0 \\ 0 & \beta_n^* s_n \end{pmatrix} - q^{2(N_n+H_n)}\sigma^z \qquad L_{\pm 2} = \tfrac{1}{2}(\sigma^z \pm I) \tag{13}$$

$$L_{+1}(n) = \sqrt{2\gamma}\begin{pmatrix} 0 & \mathrm{i}\beta_n^* \\ s_n & 0 \end{pmatrix} \qquad L_{-1}(n) = \sqrt{2\gamma}\begin{pmatrix} 0 & s_n^* \\ -\mathrm{i}\beta_n & 0 \end{pmatrix}. \tag{14}$$

The parameter $\zeta \in \mathbb{C}$ which appears in equations (8)–(10) will be thought of as the *spectral parameter*, while in continuum limit (9) is a spectral problem in L in the usual 2×2 sense (Zakharov–Shabat linear system [2]); $\sigma^{x,y,z}$ are the Pauli matrices. The compatibility condition of the two linear systems equations (8) and (9) under the isospectral condition $\partial_t \zeta = 0$ is

$$\partial_t L(\zeta|n) + L(\zeta|n)V(\zeta|n) - V(\zeta|n+1)L(\zeta|n) = 0 \tag{15}$$

and this coincides with equations (2), independent of ζ. However, $\zeta = \mathrm{e}^{\mathrm{i}\gamma\lambda}$, $\lambda \in \mathbb{C}$ as introduced in [1]; λ is a second 'spectral parameter' and the real axis in the λ-plane is the circle of unit radius in the ζ-plane; λ is the usual spectral parameter for equations (1) derived in the continuum limit. Note that time t is suppressed in equations (8) and (9): an explicit time dependence will be indicated only where and when it is needed. Reference to equations (8) and (9) may make plain the fact that the function $\Psi_n(\zeta)$ possesses essential singularities of rank

2 at $\zeta = 0, \infty$. It is also important to note that the linear equations (8) and (9) are invariant under the transformations

$$\Psi_n(\zeta) \to (-1)^{n-1}\sigma^y \Psi_n^*\left(\frac{1}{\zeta^*}\right)\sigma^y \qquad \Psi_n(\zeta) \to \sigma^z \Psi_n(-\zeta)\sigma^z. \tag{16}$$

We can now turn to the derivation of exact solutions of the LMB system equations (2). For this, as mentioned, we develop a variant of the Darboux–Bäcklund dressing procedure [11] rather then any inverse scattering method [2, 12]. The essence of the dressing procedure is to choose a 'seed' solution of the system equations (2), typically some trivial solution, and construct from it a new solution associated with additional points ζ_ν, $\nu = 1, \ldots, N$ (say) of the discrete spectrum: thus $\det \Psi_n(\zeta_\nu, t) = 0$ [2, 11, 13] for the new solution $\Psi_n(\zeta, t)$.

For initial and boundary conditions observe that for SIT and the envelope MB system equations (1), the typical experimental situation is the half-space problem: an initial optical pulse enters, supposedly without reflection from $x < 0$ into the resonant medium $x \geqslant 0$ and here breaks up into background radiation and a sequence of soliton pulses. The corresponding mathematical problem is the Cauchy problem at the point $x = 0$: $\varepsilon(x, t)|_{x=0} = \varepsilon_0(t)$ together with the asymptotic boundary conditions (in t) that for $x > 0, \mathcal{N} \to \mathcal{N}_-, \rho \to 0$ as $t \to -\infty$. For the so-called 'attenuator' N_- is the ground state $N_- = -1$ of the inversion density. For the lattice problem we therefore take the half-space problem in which $\beta_n(t)$ and $s_n(t)$ are sufficiently decreasing for $|t| \to \infty$, while $H_n(t) \to H$ such that H corresponds to N_-. In this way we would look for a solution in the half-space $n > 0$, for which it becomes the Cauchy problem specified by the conditions

$$\beta_n(t)|_{n=1} = \beta_1(t) \qquad s_n(t)|_{n=1} = s_1(t) \qquad H_n(t)|_{n=1} = H_1(t). \tag{17}$$

With this as motivation we report in this paper exact N-soliton solutions derived by the dressing procedure based on the seed solution

$$\beta_n = 0 \qquad s_n = 0 \qquad H_n = H. \tag{18}$$

The corresponding solution of the linear system equations (8) and (9) is then

$$\Psi_n^{(0)}(\zeta, t) = \exp\left\{-\tfrac{1}{4}\sigma^z t\right\}\left(\zeta^2 - \frac{1}{\zeta^2}\right)\begin{pmatrix} z^n(\zeta) & 0 \\ 0 & (-z\,(1/\zeta))^n \end{pmatrix} \tag{19}$$

where $z(\zeta) = \frac{1}{2\gamma}\left(\zeta^2 q^{-H} - q^H\right)$, while the corresponding operator $V^{(0)}(\zeta|n, t)$ has $V_0^{(0)} = V_{\pm 1}^{(0)} = 0$, and $V_{\pm 2}^{(0)} = \mp\frac{1}{4}\sigma^z$. For the N-soliton solution of equations (2) we construct the new solution $\Psi_n^{(N)}(\zeta)$ of equations (8) and (9) through the ansatz

$$\Psi_n^{(N)}(\zeta) = F(\zeta)\Psi_n^{(0)}(\zeta). \tag{20}$$

The function $F(\zeta)$ is to have poles only at the essential singularities of $\Psi_n(\zeta)$. As was indicated above these points are 0 and ∞. This suggests the ansatz

$$F(\zeta, n, t) = F_0(n, t) + \sum_{i=1}^{N} \zeta^i F_{+i}(n, t) + \zeta^{-i} F_{-i}(n, t). \tag{21}$$

It is convenient to impose the additional conditions on $F(\zeta)$ that

$$\sigma^y F^*\left(\frac{1}{\zeta^*}\right)\sigma^y = F(\zeta) \qquad \sigma^z F(-\zeta)\sigma^z = (-1)^N F(\zeta) \tag{22}$$

which are obviously compatible with the transformation equation (16). We can also normalize the matrix $F(\zeta)$ so that for each (n, t)

$$F_{-N} = \mathcal{Q}\begin{pmatrix} f(n,t) & 0 \\ 0 & f^{-1}(n,t) \end{pmatrix} \qquad F_N = \mathcal{Q}^*\begin{pmatrix} f^{-1}(n,t) & 0 \\ 0 & f(n,t) \end{pmatrix} \tag{23}$$

where the constant $\mathcal{Q}$ is independent of n and t and $f(n, t)$ is a real function. We now choose a set of N points $\{\zeta_\nu\}_{\nu=1}^N$ where $\det \Psi_n^{(N)}(\zeta)$ is to vanish. This means

$$F(\zeta_\nu)\Phi(\zeta_\nu) = F\left(\frac{1}{\zeta_\nu^*}\right)\sigma^y\Phi^*(\zeta_\nu) = 0 \tag{24}$$

where

$$\Phi(\zeta_\nu) = \begin{pmatrix} \Phi_1(\zeta_\nu) \\ \Phi_2(\zeta_\nu) \end{pmatrix} = \Psi_n^{(0)}(\zeta_\nu)\begin{pmatrix} 1 \\ -c_\nu \end{pmatrix}$$

and c_ν are constants independent of n and t. The set $\{\zeta_\nu, c_\nu\}_{\nu=1}^N$ together constitute a necessary and complete set of parameters (spectral data) for an N-soliton solution [2].

The system of equations (24) has a unique solution satisfying conditions equations (22) and (23) if we choose

$$\mathcal{Q} = \exp\left[\mathrm{i}\frac{\pi}{2}N + \mathrm{i}\sum_{\nu=1}^N \alpha_\nu\right] \qquad \zeta_\nu = \mathrm{e}^{\gamma_\nu + \mathrm{i}\alpha_\nu} \tag{25}$$

where $\gamma_\nu, \alpha_\nu \in \mathbb{R}$. In so far as $\zeta = \mathrm{e}^{\mathrm{i}\gamma\lambda} = \mathrm{e}^{\gamma_\nu + \mathrm{i}\alpha_\nu}$ and λ is the spectral parameter for equations (1) we are interested in zeros ζ_ν defined by the half λ-plane $\mathrm{Im}\,\lambda \geqslant 0$ which lie inside the circle $|\zeta| = 1$ in the ζ-plane. The linear system, (8) and (9) is invariant under the gauge transformation (20) with the potentials written as

$$\begin{aligned} &F_{-N+1}F_{-N}^{-1} = \sqrt{2\gamma}\begin{pmatrix} 0 & -s_{n-1}^* \\ -\mathrm{i}\beta_n & 0 \end{pmatrix} \\ &q^H\frac{f(n+1,t)}{f(n,t)} = q^{N_n+H_n}. \end{aligned} \tag{26}$$

We turn next to a determination of the matrices $F_{\pm i}(n, t)$. The conditions equations (22) suggest that we should take the matrices F_{-N+2k} diagonal, and the matrices $F_{-N+2k-1}$ off-diagonal in agreement with the first of equations (26).

So will F_{-N}^{-1} be diagonal from equation (23) we set

$$F_{-N}^{-1}F_{-N+2k-1} = \begin{pmatrix} 0 & y_k \\ \tilde{y}_k & 0 \end{pmatrix} \qquad F_{-N}^{-1}F_{-N+2k} = \begin{pmatrix} x_k & 0 \\ 0 & \tilde{x}_k \end{pmatrix} \tag{27}$$

in which $y_k, \tilde{y}_k, x_k, \tilde{x}_k$ are (so far) arbitrary independent complex numbers.

Then the conditions for the zeros ζ_ν equations (24) mean that we can instead solve

$$(1, 0) + \sum_{k=1}^N z_k\sigma_\nu^{2k} = 0 \qquad \nu = 1, \ldots, N \tag{28}$$

in which the z_k are row-vectors $z_k = (x_k, y_k)$, and the matrices $\sigma_\nu = S_\nu\Lambda_\nu S_\nu^{-1}$ in which $\Lambda_\nu = \mathrm{diag}(\zeta_\nu, 1/\zeta_\nu^*)$; the matrices S_ν are defined as

$$S_\nu = \begin{pmatrix} \Phi_1(\zeta_\nu) & -\Phi_2^*(\zeta_\nu) \\ \dfrac{1}{\zeta_\nu}\Phi_2(\zeta_\nu) & \zeta_\nu^*\Phi_1^*(\zeta_\nu) \end{pmatrix} \tag{29}$$

and are determined from equations (24) using the seed solution equation (18). In this way the N-soliton solution of equations (2) is put in the form

$$\beta_n = -\frac{\mathrm{i}}{\sqrt{2\gamma}} y_N^* x_N \qquad s_{n-1} = -\frac{1}{\sqrt{2\gamma}} \frac{Q^*}{Q} \frac{y_1^*}{x_N} \tag{30}$$

$$q^{-2(N_n+H_n)} = q^{-2H} \frac{x_N(n+1)}{x_N(n)} \tag{31}$$

where $x_N(n, t)$, etc is determined from equations (28).

For the one-soliton case, $N = 1$, we can choose the single point of the discrete spectrum $\zeta_0 = \mathrm{e}^{\gamma_0 + \mathrm{i}\alpha_0}$ (say) and $\gamma_0 < 0$. We then find the formulae

$$\beta_n(t) = \mathrm{i}\sqrt{\frac{2}{\gamma}} \sinh(2\gamma_0) \frac{\exp \mathrm{i}(\phi(n,t) - \alpha_0)}{\cosh(\psi(n,t) - \gamma_0)} \tag{32}$$

$$s_{n-1}(t) = -\sqrt{\frac{2}{\gamma}} \sinh(2\gamma_0) \frac{\exp \mathrm{i}(\phi(n,t) + \alpha_0)}{\cosh(\psi(n,t) + \gamma_0)} \tag{33}$$

$$q^{2(N_n+H_n)} = q^{2H} \frac{1 - \tanh(\psi(n,t) - \gamma_0)\tanh\vartheta_0}{1 - \tanh(\psi(n,t) + \gamma_0)\tanh\vartheta_0}. \tag{34}$$

Here

$$\phi(n,t) = t\cosh(2\gamma_0)\sin(2\alpha_0) - n\varrho_0 + \phi_0 \tag{35}$$

$$\psi(n,t) = t\sinh(2\gamma_0)\cos(2\alpha_0) - n\vartheta_0 + \psi_0 \tag{36}$$

$$\vartheta_0 = \frac{1}{2}\ln\frac{\sinh^2(\gamma_0 - H\gamma) + \sin^2\alpha_0}{\sinh^2(\gamma_0 + H\gamma) + \sin^2\alpha_0} + 2\gamma_0 \tag{37}$$

$$\varrho_0 = \arg\frac{\sinh(\gamma_0 - H\gamma + \mathrm{i}\alpha_0)}{\sinh(\gamma_0 + H\gamma + \mathrm{i}\alpha_0)} + 2\alpha_0 \tag{38}$$

where ϕ_0 and ψ_0 are arbitrary real constants.

The formulae for various multisoliton solutions for the lattice system (2) are too complicated to be presented in detail here. Since for the *lattice* these solutions depend explicitly on the deformation parameter $q = \mathrm{e}^{\gamma}$, these N-soliton solutions ($N = 1, 2, \ldots$) are naturally thought of as *q-deformed solitons*. In the continuum limit equation (3), $q \to 1$ and $\gamma \to 0$ and q-deformation disappear.

3. Conclusions and discussion

As was mentioned above in the case of the real (imaginary) dynamical variables and in the sharp-line limit, the MB system equations (1) is equivalent to the sine–Gordon equation. The same procedure is applicable to the LMB system which means that in the case of the reduction to the real (imaginary) dynamical variables the LMB system is, in fact, a new version of the lattice sine–Gordon equation. The dressing procedure described in this paper can be extended to this case, delivering a whole variety of solutions of the (lattice) S-G equation (solitons, breathers, etc). In so far as equations (32)–(38) form a q-deformed soliton we can use this result to gain an insight into the quantum case. One objective of the investigation of the quantum MB system must be to find out, the precise nature of, and to calculate, the 'quantum soliton' solutions. In [1, 14] we introduced and solved exactly through the quantum inverse method (up to the solutions of the Bethe equations) a quantum version of the LMB system equations (2). Since this model provides a natural and exactly solvable lattice regularization of the continuous limit quantum envelope MB (or SIT) system (and recall that the quantum sine–Gordon can be embedded in this quantum MB) it is very useful for the construction of the evolution operator and for investigating the quantum dynamics of these continuous models which have a direct physical meaning. It is known from a number of quantum models [15–17] that a 'string solution' of the Bethe equations for the quantum model corresponds, in the limit of a large number of collective excitations M, to the soliton solution of the classical counterpart of the exactly solvable quantum system. Quantum features are detected in optical solitons in fibres in [18, 19] but quantum solitons as in, for example, [20] may not necessarily mean the true solitons considered in this paper. A plausible conjecture which we will justify elsewhere is that the soliton solution equation (32) for the 'electric field' is given by the matrix element $\lim_{M\to\infty}\langle 0|C(\lambda_1)C(\lambda_2)\ldots C(\lambda_M)\beta_n^\dagger B(\lambda_1)B(\lambda_1)\ldots B(\lambda_{M-1})|0\rangle$, where $B(\lambda)$ is a creation operator for a quasiparticle and $C(\lambda) = B^\dagger(\lambda)$ is an annihilation operator. The rapidities $\{\lambda_l\}_{l=1}^M$ are roots of the Bethe equations:

$$\mathrm{e}^{2\mathrm{i}\gamma M\lambda_n}\frac{\sin^M\gamma(\lambda_l-\mathrm{i}S)}{\sin^M\gamma(\lambda_l+\mathrm{i}S)}=\prod_{j=1}^{N}\frac{\sin\gamma(\lambda_l-\lambda_j-\mathrm{i})}{\sin\gamma(\lambda_l-\lambda_j-\mathrm{i})}.$$

The operator $\beta_n^\dagger$ is the electric field operator which satisfies the q-deformed q-boson algebra analogous to the algebra equation (7). In [17] it was shown that the creation operator $B(\lambda)$ plays the role of the quantum counterpart of a Blaschke multiplier which builds the classical soliton solution [12]. The dressing operator $F(\zeta)$ equation (21) up to certain modifications has the same meaning. This indicates very well how the experience obtained in the analysis of the c-number system reported in this paper can be used in understanding the quantum case. In practice this experience helps us to trace out the formation of the classical optical soliton from a large number of quantum collective excitations, a physical problem of considerable interest.

Acknowledgments

This work has been undertaken under the Finnish Center of Excellence program. One of us (RKB) is grateful to Dr P J Caudrey for helpful discussions of the soliton solutions reported in this paper, GV was partly supported by a Russian Federation research grant RFBR no 98-01-01063.

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